Equivariant fiber Shape Category of Equivariant Continuous Maps

Maia Dzadzamia, Lela Turmanidze, Ruslan Tsinaridze

maia.dzadzamia@bsu.edu.ge, lela.turmanidze@bsu.edu.ge, ruslan.tsinaridze@bsu.edu.ge
Batumi Shota Rustaveli State University

Abstract

The shape theory of continuous maps is a new branch of classical shape theory. It is a meaningful extension of the homotopy theory of absolute neighborhood retracts for the category of maps of metrizable spaces. In this report we investigate equivariant theory of absolute (neighborhood) retracts and extensors for maps of topological $\boldsymbol{G}$-spaces from a weakly hereditary class. Here we define the notion of equivariant fiber resolution of G-map. Obtained results we use for construction the equivariant fiber shape category of $\boldsymbol{G}$-maps.

Keywords and Phrases: $\boldsymbol{G}$-map, equivariant absolute (neighborhood) retracts and extensors for $\boldsymbol{G}$-maps, equivariant fiber resolution, equivariant shape category of $\boldsymbol{G}$-maps.
extensors for $\boldsymbol{G}$-maps, equivariant fiber resolution and equivariant shape category of $\boldsymbol{G}$-maps.

1. Introduction.

The purpose of this work is to construct an equivariant fiber shape category of continuous $\mathbf{G}$-maps. For this aim we develop equivariant retracts and extensors of G-maps. The notion of retracts of maps firstly was defined by Dold [**5**]. There he considered Euklidean neighborhood retracts over a fixed space B. The absolute (neighborhood) retracts and extensors for the category of maps of metrizable spaces were introduced by Ungar [**7**] and Nepomniachy and Smirnov [**6**]. V. Baladze [**2,3**] investigated the general theory of absolute (neighborhood) retracts and extensors for maps of topological spaces from a weakly hereditary topological class. In this paper we define equivariant (neighborhood) retracts and extensors for the category $\mathbf{Mor_K}$, where **K** is equivariant weekly hereditary class. Also in this work we investigate equivariant fiber resolutions of G-maps. The theorems of the existence of equivariant fiber resolution are proved. Obtained results we use for construction the equivariant fiber shape category.

Equivariant fiber Shape Category of Continuous Maps

Let $\mathbf{K}$ be a full subcategory of the category $\mathbf{Top_G}$ and $\mathbf{Mor_K}$ is the category of morphisms of K which is a weakly hereditary class of objects of $\mathbf{Top_G}$. It means:

i. If $\mathbf{G}$-space $X \in ob(\mathbf{K})$ and $f : X \to Y$ is an equivariant homomorphism, then $Y \in ob(\mathbf{K})$;

ii. If $\mathbf{G}$-space $X \in ob(\mathbf{K})$ and A is a closed invariant subset of X, then $A \in ob(\mathbf{K})$

A $\mathbf{G}$-map $f : X \to Y$ of the category $\mathbf{Mor_K}$ is called an absolute $\mathbf{G}$-retract for the category $\mathbf{Mor_K}$ ($f \in (\mathbf{AR_G}(\mathbf{Mor_K}))$) if for each closed $\mathbf{G}$-embedding $(i;i') : f \to f' \in \mathbf{Mor_K}$ there exists a $\mathbf{G}$-retraction $(r;r') : f' \to (i;i')(f)$.

A $\mathbf{G}$-map $f \in \mathbf{Mor_K}$ is called an absolute neighborhood $\mathbf{G}$-retract for the category $\mathbf{Mor_K}$ ($f \in (\mathbf{ANR_G}\ (\mathbf{Mor_K}))$), if for every closed $\mathbf{G}$-embedding $(i;i'): f \to f' \in \mathbf{Mor_K}$ there exist a neighborhood f'' of $(i;i')(f)$ in f' and a $\mathbf{G}$-retraction $(r;r'): f'' \to (i;i')(f)$.

By $\mathbf{AR_G}(\mathbf{Mor_K})$ $(\mathbf{ANR_G}(\mathbf{Mor_K}))$ we denote the category of all absolute $\mathbf{G}$-retracts (absolute neighborhood $\mathbf{G}$-retracts) for the category $\mathbf{Mor_K}$.

By $\mathbf{H}(\mathbf{AR_G}(\mathbf{Mor_K}))(\mathbf{H}(\mathbf{ANR_G}(\mathbf{Mor_K})))$ we denote the $\mathbf{G}$-homotopy category of category $\mathbf{AR_G}(\mathbf{Mor_K})$ $(\mathbf{ANR_G}(\mathbf{Mor_K}))$.

It is clear, if $f \in (\mathbf{AR}\ (\mathbf{Mor_K}))$, then $f \in (\mathbf{ANR}\ (\mathbf{Mor_K}))$.

A $\mathbf{G}$-map $f \in (\mathbf{A(N)R}\ (\mathbf{Mor_K}))$ is also called an $\mathrm{A(N)R_G(Mor_K)}$ -map.

Definition 2. A $\mathbf{G}$-map h is called an absolute (neighborhood) extensor for the category $\mathbf{Mor_K}$ ($h \in (\mathbf{A(N)E_G}\ (\mathbf{Mor_K})))$ whenever f is a closed $\mathbf{G}$-submap of a $\mathbf{G}$-map $f' \in \mathbf{Mor_K}$, then any morphism $(\varphi;\varphi'): f \to h$ has a $\mathbf{G}$-extension $(\tilde{\varphi};\tilde{\varphi}')$ from f' (some neighborhood f'' of f in f') into h .

By $\mathbf{AE_G}(\mathbf{Mor_K})$ $(\mathbf{ANE_G}(\mathbf{Mor_K}))$ we denote the category of all absolute extensors (absolute neighborhood extensors) for the category $\mathbf{Mor_K}$.

By $\mathbf{H}(\mathbf{AE_G}(\mathbf{Mor_K}))$ $(\mathbf{H}(\mathbf{ANE_G}(\mathbf{Mor_K})))$ we denote the $\mathbf{G}$-homotopy category of category $\mathbf{AE_G}(\mathbf{Mor_K})$ $(\mathbf{ANE_G}(\mathbf{Mor_K}))$.

Clearly, if $h \in \mathbf{AE_G}(\mathbf{Mor_K})$, then $h \in (\mathbf{ANE_G}(\mathbf{Mor_K}))$.

A $\mathbf{G}$-map $h \in (\mathbf{A(N)E_G}(\mathbf{Mor_K}))$ is also called an $\mathrm{A(N)E_G(Mor_K)}$ -map.

Proposition 1. *If* M *and* N *are* $\mathrm{A(N)E}_G(\mathrm{K})$ *-spaces, then the projection* $\pi_N : M \times N \to N$ *is an* $\mathrm{A(N)E_G(Mor_K)}$ *-map.*

Proposition 2. *If* $p_\alpha : E_\alpha \to B_\alpha$, $\alpha \in \mathrm{A}$ *, is an* $\mathrm{AE_G(Mor_K)}$ *-map, then*

$$\prod_{\alpha\in A} \mathrm{p}_\alpha : \prod_{\alpha\in A} E_\alpha \to \prod_{\alpha\in A} B_\alpha$$

is an $\mathrm{AE_G(Mor_K)}$ *-map.*

Proposition 3. *Let* $p_i : E_i \to B_i; i = 1,2,....,n$ *, are* $\mathrm{A(N)E_G(Mor_K)}$ *-maps. Then*

$$\prod_{i=1}^{\mathrm{n}} p_i : \prod_{i=1}^{\mathrm{n}} E_{\mathrm{i}} \to \prod_{i=1}^{\mathrm{n}} B_{\mathrm{i}}$$

is an $\mathrm{A(N)E_G(Mor_K)}$ *-map.*

Proposition 4. *Every retract (neighborhood retract)* $p' : E' \to B'$ *of an* $\mathrm{AE_G(Mor_K)}$ *-map (*$\mathrm{A(N)E_G(Mor_K)}$ *-map)* $\mathrm{p} : E \to B$ *is an* $\mathrm{AE_G(Mor_K)}$ *-map. (*$\mathrm{A(N)E_G(Mor_K)}$ *-map).*

Let $\boldsymbol{f} = \left\{f_{\mu}, \left(p_{\mu\mu'}, p'_{\mu\mu'}\right), M\right\}$ an inverse system of the category $\boldsymbol{pro} - \boldsymbol{Mor}_{\boldsymbol{Top_G}}$ and let (f) be a rudimentary system whose term is only a *G*- map $f: X \to X'$ of a category $\mathbf{Top}_{\boldsymbol{G}}$. **Definition**

5. An equivariant fiber resolution of a G - map f is a morphism $(\boldsymbol{p},\boldsymbol{p}') = \{(p_\mu, p_{\mu'}), \mu \in M\}: (f) \to \boldsymbol{f}$ of the category $\mathbf{pro-Mor_{Top_G}}$ which for any $G-ANR(Mor_M)$ −map $t: P \to P'$ and a pair (α, α') of coverings $\alpha \in Cov(P)$ and $\alpha' \in Cov(P')$ satisfies the following two conditions:

EFR1) for every morphism $(\varphi, \varphi'): f \to t$ there exist $\mu \in M$ and a morphism $\left(\varphi_\mu, {\varphi'}_{\mu'}\right): f_\mu \to t$ such that $\left(\varphi_\mu, {\varphi'}_{\mu'}\right) \cdot (p_\mu, p_{\mu'})$ and (φ, φ') are (α, α')-near;

EFR2) there exist a pair (β, β') of coverings $\beta \in Cov(P)$ and $\beta' \in Cov(P')$ with the following property, if $\mu \in M$ and $\left(\varphi_\mu, {\varphi'}_{\mu'}\right)$, $\left(\psi_\mu, {\psi'}_{\mu'}\right): f_\mu \to t$ are morphisms such that the morphisms $\left(\varphi_\mu, {\varphi'}_{\mu'}\right) \cdot (p_\mu, p_{\mu'})$ and $\left(\psi_\mu, {\psi'}_{\mu'}\right) \cdot (p_\mu, p_{\mu'})$ are (β, β')-near, then there exists a $\mu' \geq \mu$ such that $\left(\varphi_\mu, {\varphi'}_{\mu'}\right) \cdot (p_\mu, p_{\mu'})$ and $\left(\psi_\mu, {\psi'}_{\mu'}\right) \cdot (p_\mu, p_{\mu'})$ are (α, α')-near.

If in a equivariant fiber resolution $(\boldsymbol{p},\boldsymbol{p}'): (f) \to \boldsymbol{f}$ all f_μ are $G-ANR(Mor_M)$ −maps, then $(\boldsymbol{p},\boldsymbol{p}')$ we call an $\boldsymbol{G-ANR(Mor_M)}$ - fiber resolution.

Theorem 6. Every $\boldsymbol{G}$ −map $f: X \to X'$ of topological $\boldsymbol{G}$ −spaces admits an $\boldsymbol{G-ANR(Mor_M)}$ - fiber resolution.

Lemma 7. Let $p: E \to B$ be a G-map and let $r': P' \to Q'$, $r: P \to Q$ be $G-ANR(Mor_M)$ −maps. Let $(f, f_1): p \to r'$ and $(h_0, {h_0}'), (h_1, {h_1}'): r' \to r$ be morphisms such that $(h_0, {h_0}') \cdot (f, f_1)$ and $(h_1, {h_1}') \cdot (f, f_1)$ are G-homotopic. Then there exist a $G-ANR(Mor_M)$ −map $r'': P'' \to Q''$ and morphisms $(f', {f_1}'): p \to r''$ and $(h, h'): r'' \to r'$ such that $(f, f_1) = (h, h') \cdot (f', f_1')$ and $(h_0, {h_0}') \cdot (h, h') \simeq_G (h_1, {h_1}') \cdot (h, h')$.

Definition 8. Let $f: X \to X'$ be a G-map of topological G-spaces, $[\boldsymbol{f}] = \{f_\lambda, [(p_{\lambda\lambda'}, {p'}_{\lambda\lambda'})], \Lambda\}$ be an inverse system in $\boldsymbol{H}(\mathbf{Mor_{Top_G}})$ and $[(\boldsymbol{p},\boldsymbol{p}')]: f \to [\boldsymbol{f}]$ be a morphism of pro — $\boldsymbol{H}(\mathbf{pro-Mor_{Top_G}})$ We call $[(\boldsymbol{p},\boldsymbol{p}')]$ an equivariant fiber expansion of the G-map f, provided it has the following properties:

FE1) for every $G-ANR(Mor_M)$ −map $t: P \to P'$ and a morphism $(\varphi, \varphi'): f \to t$ there is an index $\lambda \in \Lambda$ and a morphism $(\varphi_\lambda, {\varphi_\lambda}'): f_\lambda \to t$ such that $(\varphi, \varphi') \simeq_G (\varphi_\lambda, {\varphi_\lambda}') \cdot (p_\lambda, p'_\lambda)$;

FE2) $(\varphi_\lambda, {\varphi_\lambda}'), (\psi_\lambda, {\psi_\lambda}'): f_\lambda \to t$ are mophisms into a $G-ANR(Mor_M)$-map $t: P \to P'$ and $(\varphi_\lambda, {\varphi_\lambda}') \cdot (p_\lambda, p'_\lambda) \simeq_G (\psi_\lambda, {\psi_\lambda}') \cdot (p_\lambda, p'_\lambda)$, then there exists an index $\lambda' \geq \lambda$ such that $(\varphi, \varphi') \cdot (p_{\lambda\lambda'}, {p'}_{\lambda\lambda'}) \simeq_G (\psi, \psi') \cdot (p_{\lambda\lambda'}, {p'}_{\lambda\lambda'})$.

If all $f_\lambda \in G-ANR(Mor_M)$, then $[(\boldsymbol{p},\boldsymbol{p}')]: f \to [\boldsymbol{f}]$ is colled G - $ANR(Mor_M)$-fiber expansion.

Theorem 9. If morphisms $(\boldsymbol{p},\boldsymbol{p}'): f \to \boldsymbol{f}$ is an equivariant fiber resolution of a G-map $f: X \to X'$, then $[(\boldsymbol{p},\boldsymbol{p}')]: f \to [\boldsymbol{f}]$ is an equivariant fiber expansion.

Theorem 10. Let $\pi : M \to M'$ be an $G-ANR(Mor_M)$ − map and let $f: X \to X'$ be a closed submap of π. Let $\boldsymbol{f} = \{f_\lambda, (p_{\lambda\lambda'}, {p'}_{\lambda\lambda'}), \Lambda\}$ be the inverse system which consists of all neighborhoods $f_\lambda: U_\lambda \to {U'}_\lambda$ of f in π and embedding morphisms$(p_{\lambda\lambda'}, {p'}_{\lambda\lambda'}): f_{\lambda'} \to f_\lambda$ for every $\lambda \leq \lambda'$. Then the family $\{(p_\lambda, p'_\lambda), \lambda \in \Lambda\}$ consisting of equivariant embeddings $(p_\lambda, p'_\lambda): f \to f_\lambda$ is the $G-ANR(Mor_M)$ −fiber resolution.

Theorem 11. The category $\boldsymbol{H}(\mathbf{ANR}_{\mathrm{G}}(\mathbf{Mor}_{\mathrm{K}}))$ is a dense subcategory of the category $\boldsymbol{H}(\mathbf{Mor}_{\mathbf{Top}_{\mathbf{G}}})$.

The equivariant shape category $\boldsymbol{FSh_G}$ of equivariant continuous maps is an abstract shape category $\boldsymbol{Sh_{(K,L)}}$ where $\boldsymbol{K} = \boldsymbol{H}(\mathbf{Mor}_{\mathbf{Top}_{\mathbf{G}}})$ and $\boldsymbol{L} = \boldsymbol{H}(\mathbf{ANR}_{\mathrm{G}}(\mathbf{Mor}_{\mathrm{K}}))$ Objects of the equivariant fiber shape category are all equivariant continuous maps of topological G-spaces, The morphisms of $\boldsymbol{FSh_G}$ are the equivalence classes of morphisms in $\boldsymbol{pro - K}$. The morphisms of $\boldsymbol{FSh_G}$ from the map f to the map g are given by the triples $[(p,p')], [(q,q')], [(\varphi,\varphi')]$, where $[(p,p')] : f \to [\boldsymbol{f}], [(q,q')] : g \to [\boldsymbol{g}]$ are $\boldsymbol{G - ANR(Mor_M)}$-fiber expansions of f and g, respectively, and$[(\varphi,\varphi')] : [f] \to [g]$ is a morphism of $\mathbf{pro - Mor}_{\mathbf{Top}_{\mathbf{G}}}$. We can define a fiber shape morphism $(\Phi,\Phi') : f \to g$ one chooses $\boldsymbol{H}(\mathbf{ANR}_{\mathrm{G}}(\mathbf{Mor}_{\mathrm{K}}))$-fiber resolutions $(p,p') : f \to [\boldsymbol{f}]$ and $(q,q') : g \to [\boldsymbol{g}]$, which exist by Theorem 9, and one chooses a morphism $[(\varphi,\varphi')]: [\boldsymbol{f}] \to [\boldsymbol{g}]$ of $\mathbf{pro - Mor}_{\mathbf{Top}_{\mathbf{G}}}$.

By $FS_G((\varphi,\varphi'))$ we denote the class of the morphism $[(\varphi,\varphi')] \in pro - (\boldsymbol{G - ANR(Mor_M)})([f],[g])$.

Let $FS_G(f) = f$ for every $f \in \boldsymbol{H}(\mathbf{Mor}_{\mathbf{Top}_{\mathbf{G}}})$. We obtain the fiber shape functor

$$FS_G: \boldsymbol{H}\left(\mathbf{Mor}_{\mathbf{Top}_{\mathbf{G}}}\right) \to \boldsymbol{FSh_G}.$$

We say that the continuous maps f and g have the same fiber shape, and we write $FS_G(f)$= FS_G(h) provided they are isomorphic objects of the fiber shape category $\boldsymbol{FSh_G}$ If f and g are equivalent objects of the category $\boldsymbol{H}(\mathbf{Mor}_{\mathbf{Top}_{\mathbf{G}}})$), then $FS_G(f)$= FS_G(h) .

Corollary 12. The fiber shape functor FS_G induces an isomorphism between $\boldsymbol{G - ANR(Mor_M)}$ and the full subcategory of the fiber shape category restricted to objects of $\boldsymbol{H(ANR(Mor_M))}$.

References:

1. S. A. Antonian and S. Mardešic´, Equivariant shape. Fund. Math., 127 (1987), 213–224.
2. V.Baladze, On shape theory for fibrations, Bull. Georgian Acad. Sci., 129(1988), 269-272.
3. V.Baladze, Fiber shape theory and resolutions, Zb. Rad. Filoz. Fak. Nisu, Ser. Mat., 5(1991), 97-107.
4. V.Baladze, Fiber shape theory, Proc. A. Razmadze Math. Inst., 132(2003), 1-70.
5. A. Dold, The fixed index of fibre-demanding maps. Invert. Math. 25(1974), 281-289.
6. G.M. Nepomniachy, Yu.M. Smirnov, On retraction of mappings. (Russian) Chechoslovak math. J.104(1979), No.29,366-377.
7. G. S. Ungar, ANR's and NES's in the category of mapping of metric spaces. Fund. Math. 95(1977), 111-127.