# On Lebesgue-Fejer-Steinhaus theorem

**Mzevinar Bakuridze[1], Vaja Tarieladze[2]**

[1]Batumi Shota Rustaveli State University

[2]Muskhelishvili Institute of Computational Mathematics, GTU

mzevinar.bakuridze@bsu.edu.ge, v.tarieladze@gtu.ge

## Abstract

In this survey, dedicated to 150-th birthday anniversary of Henri Leon Lebesgue (June 28, 1875 - July 26, 1941), in particular, we plan to demonstrate that the name "Lebesgue-Fejer-Steinhaus theorem" is justified for the following result:

**Theorem.** There exists a continuous function whose Fourier series converges point-wise, but not uniformly.

At the end we formulate a conjecture related with this theorem.

### 1. Zygmund's version

In Antony Zygmund, Trigonometric series, Third Edition, Volumes I & II Combined, with a foreword by Robert Fefferman. Cambridge Mathematical library. Cambridge University Press, 2002.

one can find the following theorem:

**Theorem 1** [1, Ch. VIII, Theorem 1.13, p.300] There exists a continuous function whose Fourier series converges point-wise, but not uniformly.

In connection of this theorem A. Zygmund refers the papers;

**Leopold Fejer**, Sur les singularites de la serie de Fourier des fonctions continues.

Annales scientifiques de l' Ecole Normale Superieure 28 (1911), 63-104. and

**Hugo Steinhaus**, Sur la convergence non-uniforme des series de Fourier, Publ. l' Ac. de Cracovie (1913), 145-160

### 1.Fichtengoltz's version

On p. 497 of

**G. M. Fichtengoltz**, Course of Differential and Integral Calculus, vol.

III (in Russian), Moscow, 1969

it is written:

" In 1906 H. Lebesgue has constructed an example of a continuous func?tion, for which its Fourier series converges to this function everywhere, but not uniformly."

Therefore, according to Fichtengoltz (June 5, 1888 - June 26, 1959),

**Theorem 1 was obtained by Lebesgue before Fejer and Stein- haus.**

Surely, Fichtengoltz meant the paper **Henri Lebesgue**, Sur la divergence et la convergence non-uniforme des series de Fourier. C. R. 141, 875-877 (1906).

### Bari's version

On p. 128 of **Bari**, Trigonometric series (in Russian), Moscow, 1961. (see beginning of §43 of Chapter I). it is written:

”We want to show that if on a continuous function f(x) we do not impose some further restrictions, then it may happen that its Fourier series may di- verge on some point, or may converge everywhere, but converge non? Uniformly near of a point. The first examples of these kind were given by Du Bois Rey- mond and Lebesgue, that’s why these facts are named as peculiarity of Du Bois Reymond (the case of divergence) and peculiarity of Lebesgue (the case of non-uniform convergence).”

We note that a reference to 1873’s German paper by Paul David Gustav du Bois-Reymond (2 December 1831 - 7 April 1889) is given, while no a reference to a Lebesgue’s work is given.

Note also that §44 of Chapter I in Bari’s book is entitled as follows:

„A continuous function with the Fourier series, which converges every- where but not uniformly“.

The reference list of Bari’s book (as well as of Zygmund’s book) does not contain H. Lebesgue’s above mentioned paper of 1906.

**Lebesgue’s role**

In the reference list of Bari’s book (as well as of Zygmund’s book) presents the following text:

**Lebesgue, H**. Lecons sur les series trigonometriques. Paris, Gautier Villars, 1906, 128 p.

§47 (p.88) of Lebesgue’s text begins as follows:

"Existence de fonctions continues representables par leurs series de Fourier non uniformement convergentes".

Because of Lebesgue’s works [6,7] we think that Theorem 1 should be named as ”The Lebesgue-Fejer-Steinhaus theorem”.

**Final remarks**

On p.89 of §47 of Lebesgue’s text [7] we can find something very interesting (we hope that our translation from French is correct):

**Lebesgue’s Question 1:** ”Is there a continuous function, which has everywhere divergent Fourier series?”

**Lebesgue’s Question 2:** "Is there a continuous function with every- where convergent Fourier series, but there is no an interval on which it would converge uniformly?"

A negative answer to the first question remained open until 1966, when Swedish mathematician Lennart Axel Edvard Carleson (born 18 March 1928) has published a proof of Lusin’s conjecture. Now we can add that, simulta- neously, L. Carleson has answered also Lebesgue’s Question 1.

After formulating the above questions, on p.89 of §47 of Lebesgue’s text [7] we can find also the following footnote:

”Dans un note de Comptes rendus (29 decembre 1902) M. Stekloff a in- dique que la reponse a la premiere de ces questions etait negative. La demon- stration de cette propriete n’a pas encore ete publiee et et les ren?seignements que contient la Note citee sont insuficants, a ce qu’il me semble, pur perme- ttre la reconstitution de cette demonstration”.

Unfortunately, we could not find the ’Note de M. Stekloff’. We can con- clude that, according to Lebesgue, in 1902 V. A. Steklov (9 January 1864 - 30 May 1926) made an attempt to prove that the Fourier series of a continuous function always has a point of convergence.

The second question posed by Lebesgue was solved positively in the above cited papers by Fejer and Steinhaus. This solution proposed by Fejer is in- cluded in in Zygmund’s and Bari’s books. Steinhaus’s paper has few citations, among them is a nice survey paper by Levan Zhizhiashvili (1934- 2006):

**L. V. Zhizhiashvili**, Some problems in the theory of simple and multiple trigonometric and orthogonal series, Uspekhi Mat. Nauk, 1973, Volume 28, Issue 2(170), 65-119.

In August 19, 2018 with the help of Professor Zbigniew Lipecki (Wro- claw, Poland) we got a scanned copy of Steinhaus' paper [8]. According to Steinhaus, Lebesgue's question 1 was initially posed by Du Bois Reymond.

As far as we could clarify until now the following version of Theorem 1 was obtained:

**Theorem 2.** There exists a continuous odd function whose Fourier series converges point-wise, but not uniformly.

Based on [1] in [2] we conjectured that if for a continuous periodic even function its Fourier series converges point-wise, then it converges uniformly (at least on some interval).

## References:


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