Generated Sets of the Complete Semigroup Binary Relations Defined by Semilattices of the Class $\Sigma_6(X,5)$

Nino Tsinaridze, Giuli Tavdgiridze, Omar Givradze

Batumi Shota Rustaveli State University

n.tsinaridze@bsu.edu.ge, g.tavdgiridze@bsu.edu.ge, omar.givradze@bsu.edu.ge

ABSTRACT. In this article, we study generated sets of the complete semigroups binary relations defined by $X-$semilattices unions of the class $\Sigma_6(X,5)$.

Key words: Semigroup, semilattice, binary relation.

1. Introduction.

Let X be an arbitrary nonempty set, D is an $X-$semilattice of unions which closed with respect to the set-theoretic union of elements from D, f be an arbitrary mapping of the set X in the set D. To each mapping f we put into correspondence a binary relation α_f on the set X that satisfies the condition $\alpha_f=\bigcup_{x\in X}(\{x\}\times f(x))$. The set of all such α_f ($f:X\to D$) is denoted by $B_X(D)$. It is easy to prove that $B_X(D)$ is a semigroup with respect to the operation of multiplication of binary relations, which is called a complete semigroup of binary relations defined by an $X-$semilattice of unions D.

We denote by $\varnothing$ an empty binary relation or an empty subset of the set X. The condition $(x,y)\in\alpha$ will be written in the form $x\alpha y$. Further, let $x,y\in X$, $Y\subseteq X$, $\alpha\in B_X(D)$, $\breve{D}=\bigcup_{Y\in D}Y$ and $T\in D$. We denote by the symbols $y\alpha$, $Y\alpha$, $V(D,\alpha)$, X^* and $V(X^*,\alpha)$ the following sets:

$$y\alpha=\{x\in X\mid y\alpha x\},\ Y\alpha=\bigcup_{y\in Y}y\alpha,\ V(D,\alpha)=\{Y\alpha\mid Y\in D\},\ X^*=\{Y\mid\varnothing\neq Y\subseteq X\},$$
$$V(X^*,\alpha)=\{Y\alpha\mid\varnothing\neq Y\subseteq X\},\ D_T=\{Z\in D\mid T\subseteq Z\},\ Y_T^\alpha=\{y\in X\mid y\alpha=T\}.$$

Definition 1. *We say that an element* α *of the semigroup* $B_X(D)$ *is external if* $\alpha\neq\delta\circ\beta$ *for all* $\delta,\beta\in B_X(D)\setminus\{\alpha\}$ (see [1], Definition 1.15.1).

It is well know, that if B is all external elements of the semigroup $B_X(D)$ and B' be any generated set for the $B_X(D)$, then $B\subseteq B'$ (see [1], Lemma 1.15.1).

2. Related Work.

2. Let $\Sigma_6(X,5)$ be a class of all $X-$semilattices of unions whose every element is isomorphic to an $X-$semilattice of unions $D=\{Z_4,Z_3,Z_2,Z_1,\breve{D}\}$, which satisfies the condition:

$$Z_4\subset Z_1\subset\breve{D},\ Z_3\subset Z_1\subset\breve{D},\ Z_2\subset\breve{D},\ Z_4\setminus Z_3\neq\varnothing,\ Z_3\setminus Z_4\neq\varnothing,$$
$$Z_2\setminus Z_1\neq\varnothing,\ Z_1\setminus Z_2\neq\varnothing,\ Z_4\cup Z_3=Z_1,\ Z_4\cup Z_2=Z_1\cup Z_2=Z_1\cup Z_2=\breve{D}.$$

(see Fig. 1).

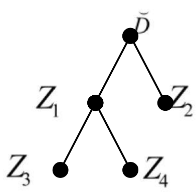

Fig. 1

It is easy to see that $\tilde{D}=\{Z_4,Z_3,Z_2\}$ is irreducible generating set of the semilattice D.

Let $C(D)=\{P_0,P_1,P_2,P_3,P_4\}$ is a family of sets, where P_0,P_1,P_2,P_3,P_4 are pairwise disjoint subsets of the set X and $\varphi=\begin{pmatrix}\breve{D} & Z_1 & Z_2 & Z_3 & Z_4\\ P_0 & P_1 & P_2 & P_3 & P_4\end{pmatrix}$ is a mapping of the semilattice D onto the family of sets $C(D)$. Then the formal equalities of the semilattice D has a form:

$$\begin{aligned}\breve{D}&=P_0\cup P_1\cup P_2\cup P_3\cup P_4,\\ Z_1&=P_0\cup P_2\cup P_3\cup P_4,\\ Z_2&=P_0\cup P_1\cup P_3\cup P_4,\\ Z_3&=P_0\cup P_2\cup P_4,\\ Z_4&=P_0\cup P_2\cup P_3.\end{aligned}\qquad(1)$$

here the elements P_4,P_3,P_2,P_1 are basis sources, the element P_0 are sources of completeness of the semilattice D. Therefore $|X|\geq 4$ since $|P_4|\geq 1$, $|P_3|\geq 1$, $|P_2|\geq 1$, $|P_1|\geq 1$ (see [1], chapter 11).

From the formal equalities of the semilattise D immediately follows, that:

$$\begin{aligned}&P_4=Z_3\setminus Z_4,\ P_3=Z_4\setminus Z_3,\ P_3=Z_1\setminus Z_3,\\ &P_2=Z_1\setminus Z_2,\ P_1=Z_2\setminus Z_1,\ P_0=Z_4\cap Z_3\cap Z_2.\end{aligned}\qquad(2)$$

In the sequel, by symbol $\Sigma_{6.0}(X,5)$ we denoted all semilattices $D=\{Z_4,Z_3,Z_2,Z_1,\breve{D}\}$ of the class $\Sigma_6(X,5)$ for which $Z_4\cap Z_3\cap Z_2\neq\varnothing$. Of the last inequality from the formal equalities (1) of a semilattise D follows that $Z_4\cap Z_3\cap Z_2=P_0\neq\varnothing$, i.e. $|X|\geq 5$ since $P_4\neq\varnothing$, $P_3\neq\varnothing$, $P_2\neq\varnothing$, $P_1\neq\varnothing$, $P_0\neq\varnothing$.

Let $D\in\Sigma_{6.0}(X,5)$. By symbols A_4,A_3,A_2 and A_1 we denoted the following sets:

$$\begin{aligned}A_4&=\{\{Z_4,Z_3,Z_1,\breve{D}\},\{Z_4,Z_2,Z_1,\breve{D}\},\{Z_3,Z_2,Z_1,\breve{D}\}\},\\ A_3&=\{\{Z_4,Z_3,Z_1\},\{Z_4,Z_2,\breve{D}\},\{Z_3,Z_2,\breve{D}\},\{Z_4,Z_1,\breve{D}\},\{Z_3,Z_1,\breve{D}\},\{Z_2,Z_1,\breve{D}\}\},\\ A_2&=\{\{Z_4,Z_1\},\{Z_3,Z_1\},\{Z_4,\breve{D}\},\{Z_3,\breve{D}\},\{Z_1,\breve{D}\},\{Z_2,\breve{D}\}\},\\ A_1&=\{\{Z_4\},\{Z_3\},\{Z_2\},\{Z_1\},\{\breve{D}\}\}.\end{aligned}$$

Lemma 2.1. *Let $D\in\Sigma_{6.0}(X,5)$. $\alpha=\delta\circ\beta$ for some $\alpha,\delta,\beta\in B_X(D)$. Then the following statements are true:*

a) *Let $T,T'\in\{Z_4,Z_3,Z_2\}$, $T\neq T'$. if $T,T'\in V(D,\alpha)$, then α is external element of the semigroup $B_X(D)$;*

b) *If $Z_2,Z_1\in V(D,\alpha)$, then α is external element of the semigroup $B_X(D)$.*

Let $D\in\Sigma_{6.0}(X,5)$. By symbols A_0, $B(A_0)$ and B_0 we denoted the following sets:

$$\begin{aligned}&A_0=\{\{Z_4,Z_3,Z_1,\breve{D}\},\{Z_4,Z_2,Z_1,\breve{D}\},\{Z_3,Z_2,Z_1,\breve{D}\},\{Z_4,Z_3,Z_1\},\{Z_4,Z_2,\breve{D}\},\{Z_3,Z_2,\breve{D}\},\{Z_2,Z_1,\breve{D}\}\},\\ &B(A_0)=\{\alpha\in B_X(D)\mid V(X^*,\alpha)\in A_0\};\ B_0=\{\alpha\in B_X(D)\mid V(X^*,\alpha)=D\}.\end{aligned}$$

Remark, that the of the sets B_0 and $B(A_0)$ are external elements for the semigroup $B_X(D)$.

Lemma 2.2. *Let $D\in\Sigma_{6.0}(X,5)$. Then the following statements are true:*

a) *if quasinormal representation of a binary relation α has a form*

$$\alpha=(Y_4^\alpha\times Z_4)\cup(Y_1^\alpha\times Z_1)\cup(Y_0^\alpha\times\breve{D}),$$

where $Y_4^\alpha,Y_1^\alpha,Y_0^\alpha\notin\{\varnothing\}$, then α is generating by elements of the elements of set $B(A_0)$;

b) *if quasinormal representation of a binary relation α has a form*

$$\alpha=(Y_3^\alpha\times Z_3)\cup(Y_1^\alpha\times Z_1)\cup(Y_0^\alpha\times\breve{D}),$$

where $Y_3^\alpha,Y_1^\alpha,Y_0^\alpha\notin\{\varnothing\}$, then α is generating by elements of the elements of set $B(A_0)$;

Lemma 2.3. *Let $D\in\Sigma_{6.0}(X,5)$. Then the following statements are true:*

a) *if quasinormal representation of a binary relation has a form $\alpha=(Y_4^\alpha\times Z_4)\cup(Y_1^\alpha\times Z_1)$,* *where* $Y_4^\alpha, Y_1^\alpha \notin \{\varnothing\}$ *, then* α *is generating by elements of the elements of set* $B(A_0)$ *;*

b) *if quasinormal representation of a binary relation has a form* $\alpha = (Y_3^\alpha \times Z_3) \cup (Y_1^\alpha \times Z_1)$,

where $Y_3^\alpha, Y_1^\alpha \notin \{\varnothing\}$ *, then* α *is generating by elements of the elements of set* $B(A_0)$ *;*

c) *if quasinormal representation of a binary relation has a form* $\alpha = (Y_4^\alpha \times Z_4) \cup (Y_0^\alpha \times \breve{D})$,

where $Y_4^\alpha, Y_0^\alpha \notin \{\varnothing\}$ *, then* α *is generating by elements of the elements of set* $B(A_0)$ *;*

d) *if quasinormal representation of a binary relation has a form* $\alpha = (Y_3^\alpha \times Z_3) \cup (Y_0^\alpha \times \breve{D})$,

where $Y_3^\alpha, Y_0^\alpha \notin \{\varnothing\}$ *, then* α *is generating by elements of the elements of set* $B(A_0)$ *;*

e) *if quasinormal representation of a binary relation has a form* $\alpha = (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D})$,

where $Y_1^\alpha, Y_0^\alpha \notin \{\varnothing\}$ *, then* α *is generating by elements of the elements of set* $B(A_0)$ *;*

f) *if quasinormal representation of a binary relation has a form* $\alpha = (Y_2^\alpha \times Z_2) \cup (Y_0^\alpha \times \breve{D})$,

where $Y_2^\alpha, Y_0^\alpha \notin \{\varnothing\}$ *, then* α *is generating by elements of the elements of set* $B(A_0)$.

g) *if quasinormal representation of a binary relation has a form* $\alpha = X \times Z_1$*, then* α *is generating by elements of the elements of set* $B(A_0)$.

h) *if quasinormal representation of a binary relation has a form* $\alpha = X \times \breve{D}$*, then* α *is generating by elements of the elements of set* $B(A_0)$.

Lemma 2.4 *. Let* $D \in \Sigma_{6.0}(X,5)$ *. Then the following statements is true:*

a) *if* $|X \setminus \breve{D}| \geq 1$ *and* $T \in \{Z_4, Z_3, Z_2\}$ *, then binary relation* $\alpha = X \times T$ *is generating by elements of the elements of set* $B(A_0)$ *;*

b) *if* $X = \breve{D}$ *and* $T \in \{Z_4, Z_3, Z_2\}$*, then binary relation* $\alpha = X \times T$ *is external element for the semigroup* $B_X(D)$.

3. Result

Theorem 2.1. *Let* $D \in \Sigma_{6.0}(X,5)$ *and*

$$A_0 = \{\{Z_4, Z_3, Z_1, \breve{D}\}, \{Z_4, Z_2, Z_1, \breve{D}\}, \{Z_3, Z_2, Z_1, \breve{D}\}, \{Z_4, Z_3, Z_1\}, \{Z_4, Z_2, \breve{D}\}, \{Z_3, Z_2, \breve{D}\}, \{Z_2, Z_1, \breve{D}\}\},$$
$$B(A_0) = \{\alpha \in B_X(D) \mid V(D,\alpha) \in A_0\};\ B_0 = \{\alpha \in B_X(D) \mid V(X^*,\alpha) = D\}.$$

Then the following statements are true:

a) *if* $|X \setminus \breve{D}| \geq 1$ *, then the* $S_0 = B_0 \cup B(A_0)$ *is irreducible generating set for the semigroup* $B_X(D)$ *;*

b) *if* $X = \breve{D}$ *, then the* $S_1 = B_0 \cup B(A_0) \cup \{X \times Z_4, X \times Z_3, X \times Z_2\}$ *is irreducible generating set for the semigroup* $B_X(D)$ *.*

Proof. Let $D \in \Sigma_{6.0}(X,5)$ and $|X \setminus \breve{D}| \geq 1$. First, we proved that every element of the semigroup $B_X(D)$ is generating by elements of the set S_0. Indeed, let α be arbitrary element of the semigroup $B_X(D)$. Then quasinormal representation of a binary relation α has a form

$$\alpha = (Y_4^\alpha \times Z_4) \cup (Y_3^\alpha \times Z_3) \cup (Y_2^\alpha \times Z_2) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D}),$$

where $Y_4^\alpha \cup Y_3^\alpha \cup Y_2^\alpha \cup Y_1^\alpha \cup Y_0^\alpha = X$ and $Y_i^\alpha \cap Y_j^\alpha = \varnothing$ $(0 \leq i \neq j \leq 4)$. For the $|V(X^*,\alpha)|$ we consider the following cases:

1) $|V(X^*,\alpha)| = 5$. Then $\alpha \in B_0$ and $B_0 \subset S_0$ by definition of a set S_0.

2) $\left|V\left(X^*,\alpha\right)\right|=4$. Then $V\left(X^*,\alpha\right)\in A_4=\left\{\left\{Z_4,Z_3,Z_1,\breve{D}\right\},\left\{Z_4,Z_2,Z_1,\breve{D}\right\},\left\{Z_3,Z_2,Z_1,\breve{D}\right\}\right\}\subset A_0$ i.e. $\alpha\in B(A_0)$ and $B(A_0)\subset S_0$ by definition of a set S_0.

3) $\left|V\left(X^*,\alpha\right)\right|=3$. Then we have

$$V\left(X^*,\alpha\right)\in A_3=\left\{\left\{Z_4,Z_3,Z_1\right\},\left\{Z_4,Z_2,\breve{D}\right\},\left\{Z_3,Z_2,\breve{D}\right\},\left\{Z_4,Z_1,\breve{D}\right\},\left\{Z_3,Z_1,\breve{D}\right\},\left\{Z_2,Z_1,\breve{D}\right\}\right\}.$$

By definition of e set A_0 we have that $\left\{\left\{Z_4,Z_3,Z_1\right\},\left\{Z_4,Z_2,\breve{D}\right\},\left\{Z_3,Z_2,\breve{D}\right\},\left\{Z_2,Z_1,\breve{D}\right\}\right\}\subset A_0$, i.e. in this case $\alpha\in B(A_0)$ and $B(A_0)\subset S_0$ by definition of a set S_0.

If $V\left(X^*,\alpha\right)\in\left\{\left\{Z_4,Z_1,\breve{D}\right\},\left\{Z_3,Z_1,\breve{D}\right\}\right\}$, then from the statement $a)$ and $b)$ of the Lemma 2.2 element α is generating by elements $B(A_0)$ and $B(A_0)\subset S_0$ by definition of a set S_0.

4) $\left|V\left(X^*,\alpha\right)\right|=2$. Then we have, that

$$V\left(X^*,\alpha\right)\in A_2=\left\{\left\{Z_4,Z_1\right\},\left\{Z_3,Z_1\right\},\left\{Z_4,\breve{D}\right\},\left\{Z_3,\breve{D}\right\},\left\{Z_1,\breve{D}\right\},\left\{Z_2,\breve{D}\right\}\right\}.$$

Then from the statement $a)-f)$ of the Lemma 2.3 element α is generating by elements $B(A_0)$ and $B(A_0)\subset S_0$ by definition of a set S_0.

5) $\left|V\left(X^*,\alpha\right)\right|=1$. Then we have, that $V\left(X^*,\alpha\right)\in A_1=\left\{\left\{Z_1\right\},\left\{\breve{D}\right\},\left\{Z_4\right\},\left\{Z_3\right\},\left\{Z_2\right\}\right\}$.

If $V\left(X^*,\alpha\right)\in\left\{\left\{Z_1\right\},\left\{\breve{D}\right\}\right\}$, then from the statement $g)$, $h)$ of the Lemma 2.3 element α is generating by elements $B(A_0)$ and $B(A_0)\subset S_0$ by definition of a set S_0.

If $V\left(X^*,\alpha\right)\in\left\{\left\{Z_4\right\},\left\{Z_3\right\},\left\{Z_2\right\}\right\}$, then from the statement $a)$ of the Lemma 2.4 element α is generating by elements $B(A_0)$ and $B(A_0)\subset S_0$ by definition of a set S_0.

Thus, we have that S_0 is generating set for the semigroup $B_X(D)$.

If $\left|X\setminus\breve{D}\right|\geq 1$, then the set S_0 is irreducible generating set for the semigroup $B_X(D)$ since S_0 is a set external elements of the semigroup $B_X(D)$.

The statement $a)$ of the Theorem 2.1 is proved.

Now, let $D\in\Sigma_{6.0}(X,5)$ and $X=\breve{D}$. First, we proved that every element of the semigroup $B_X(D)$ is generating by elements of the set S_1. The cases $1)$, $2)$, $3)$ and $4)$ are proved analogously of the cases $1)$, $2)$, $3)$ and $4)$ given above and consider case, when

$$V\left(X^*,\alpha\right)\in A_1=\left\{\left\{Z_1\right\},\left\{\breve{D}\right\},\left\{Z_4\right\},\left\{Z_3\right\},\left\{Z_2\right\}\right\}.$$

If $V\left(X^*,\alpha\right)\in\left\{\left\{Z_1\right\},\left\{\breve{D}\right\}\right\}$, then from the statement $g)$, $h)$ of the Lemma 2.3 element α is generating by elements $B(A_0)$ and $B(A_0)\subset S_0$ by definition of a set S_1.

If $V\left(X^*,\alpha\right)\in\left\{\left\{Z_4\right\},\left\{Z_3\right\},\left\{Z_2\right\}\right\}$, then $\alpha\in S_1$ by definition of a set S_1.

Thus, we have that S_1 is generating set for the semigroup $B_X(D)$.

If $X=\breve{D}$, then the set S_1 is irreducible generating set for the semigroup $B_X(D)$ since S_1 is a set external elements of the semigroup $B_X(D)$.

The statement $b)$ of the Theorem 2.1 is proved.

Theorem 2.1 is proved.

Theorem 2.2. *Let* $D=\left\{Z_4,Z_3,Z_2,Z_1,\breve{D}\right\}\in\Sigma_{6.0}(X,5)$ *and*

$$\mathcal{A}_0=\left\{\left\{Z_4,Z_3,Z_1,\breve{D}\right\},\left\{Z_4,Z_2,Z_1,\breve{D}\right\},\left\{Z_3,Z_2,Z_1,\breve{D}\right\},\left\{Z_4,Z_3,Z_1\right\},\left\{Z_4,Z_2,\breve{D}\right\},\left\{Z_3,Z_2,\breve{D}\right\},\left\{Z_2,Z_1,\breve{D}\right\}\right\},$$
$$B(\mathcal{A}_0)=\left\{\alpha\in B_X(D)\,|\,V(D,\alpha)\in\mathcal{A}_0\right\};\ B_0=\left\{\alpha\in B_X(D)\,|\,V\left(X^*,\alpha\right)=D\right\}.$$

Then the following statements are true:

a) *If* $\left|X\setminus\breve{D}\right|\ge 1$ *, then the number* $|S_0|$ *elements of the set* $S_0=B_0\cup B(\mathcal{A}_0)$ *is equal to*

$$|S_0|=5^n-2\cdot 3^n+5\,.$$

b) *If* $X=\breve{D}$ *, then the number* $|S_1|$ *elements of the set* $S_1=B_0\cup B(\mathcal{A}_0)\cup\{X\times Z_4,X\times Z_3,X\times Z_2\}$ *is equal to*

$$|S_1|=5^n-2\cdot 3^n+8\,.$$

Proof. Let number of a set X is equal to n, i.e. $|X|=n$. Let $S_n=\{\varphi_1,\varphi_2,...,\varphi_{n!}\}$ is a group all one to one mapping of a set $M=\{1,2,...,n\}$ on the set M and $\varphi_{i_1},\varphi_{i_2},...,\varphi_{i_m}$ $(m\le n)$ are arbitrary elements of the group S_n, $Y_{\varphi_1},Y_{\varphi_2},...,Y_{\varphi_m}$ are arbitrary partitioning of a set X. By symbol k_n^m we denote the number elements of a set $\left\{Y_{\varphi_1},Y_{\varphi_2},...,Y_{\varphi_m}\right\}$. It is well know, that

$$k_n^m=\sum_{i=1}^{m}\frac{(-1)^{m+i}}{(i-1)!\,\cdot(m-i)!}\cdot i^{n-1}\,.$$

If $m=2,3,4,5$, then we have

$$k_n^2=2^{n-1}-1,\ k_n^3=\frac{1}{2}\cdot 3^{n-1}-2^{n-1}+\frac{1}{2},\ k_n^4=\frac{1}{6}\cdot 4^{n-1}-\frac{1}{2}\cdot 3^{n-1}+\frac{1}{2}\cdot 2^{n-1}-\frac{1}{6},$$
$$k_n^5=\frac{1}{24}\cdot 5^{n-1}-\frac{1}{6}\cdot 4^{n-1}+\frac{1}{4}\cdot 3^{n-1}-\frac{1}{6}\cdot 2^{n-1}+\frac{1}{24}.$$

If $Y_{\varphi_1},Y_{\varphi_2}$ are any two elements partitioning of a set X and $\overline{\beta}=\left(Y_{\varphi_1}\times T_1\right)\cup\left(Y_{\varphi_2}\times T_2\right)$, where $T_1,T_2\in D$ and $T_1\ne T_2$. Then number of different binary relations $\overline{\beta}$ of a semigroup $B_X(D)$ is equal to

$$2\cdot k_n^2=2^n-2\,. \tag{3}$$

If $Y_{\varphi_1},Y_{\varphi_2},Y_{\varphi_3}$ are any tree elements partitioning of a set X and

$$\overline{\beta}=\left(Y_{\varphi_1}\times T_1\right)\cup\left(Y_{\varphi_2}\times T_2\right)\cup\left(Y_{\varphi_3}\times T_3\right),$$

where T_1,T_2,T_3 are pairwise different elements of a given semilattice D. Then number of different binary relations $\overline{\beta}$ of a semigroup $B_X(D)$ is equal to

$$6\cdot k_n^3=3^n-3\cdot 2^n+3\,. \tag{4}$$

If $Y_{\varphi_1},Y_{\varphi_2},Y_{\varphi_3},Y_{\varphi_4}$ are any four elements partitioning of a set X and

$$\overline{\beta}=\left(Y_{\varphi_1}\times T_1\right)\cup\left(Y_{\varphi_2}\times T_2\right)\cup\left(Y_{\varphi_3}\times T_3\right)\cup\left(Y_{\varphi_4}\times T_4\right),$$

where T_1,T_2,T_3,T_4 are pairwise different elements of a given semilattice D. Then number of different binary relations $\overline{\beta}$ of a semigroup $B_X(D)$ is equal to

$$24\cdot k_n^4=4^n-4\cdot 3^n+3\cdot 2^{n+1}-4\,. \tag{5}$$

If $Y_{\varphi_1},Y_{\varphi_2},Y_{\varphi_3},Y_{\varphi_4},Y_{\varphi_5}$ are any four elements partitioning of a set X and

$$\overline{\beta}=\left(Y_{\varphi_1}\times T_1\right)\cup\left(Y_{\varphi_2}\times T_2\right)\cup\left(Y_{\varphi_3}\times T_3\right)\cup\left(Y_{\varphi_4}\times T_4\right)\cup\left(Y_{\varphi_5}\times T_5\right),$$

where T_1,T_2,T_3,T_4,T_5 are pairwise different elements of a given semilattice D. Then number of different binary relations $\overline{\beta}$ of a semigroup $B_X(D)$ is equal to

$$120\cdot k_n^5=5^n-5\cdot 4^n+10\cdot 3^n-10\cdot 2^n+5. \tag{6}$$

If $\alpha \in B_0$, then quasinormal representation of a binary relation α has a form

$$\alpha = \left(Y_4^\alpha \times Z_4\right) \cup \left(Y_3^\alpha \times Z_3\right) \cup \left(Y_2^\alpha \times Z_2\right) \cup \left(Y_1^\alpha \times Z_1\right) \cup \left(Y_0^\alpha \times \breve{D}\right),$$

where $Y_4^\alpha, Y_3^\alpha, Y_2^\alpha \notin \{\varnothing\}$, or a system $Y_4^\alpha, Y_3^\alpha, Y_2^\alpha, Y_1^\alpha$, or a system $Y_4^\alpha, Y_3^\alpha, Y_2^\alpha, Y_0^\alpha$, or a system $Y_4^\alpha, Y_3^\alpha, Y_2^\alpha, Y_1^\alpha, Y_0^\alpha$ are partitioning of the set X.

If the system $Y_4^\alpha, Y_3^\alpha, Y_2^\alpha$, or a system $Y_4^\alpha, Y_3^\alpha, Y_2^\alpha, Y_1^\alpha$, or a system $Y_4^\alpha, Y_3^\alpha, Y_2^\alpha, Y_0^\alpha$, or a system $Y_4^\alpha, Y_3^\alpha, Y_2^\alpha, Y_1^\alpha, Y_0^\alpha$ are partitioning of the set X. Of this from the equalities (4), (5) and (6) follows that

$$\begin{aligned}|B_0| &= \left(5^n - 5\cdot 4^n + 10\cdot 3^n - 10\cdot 2^n + 5\right) + 2\cdot\left(4^n - 4\cdot 3^n + 6\cdot 2^n - 4\right) + \\ &+ \left(3^n - 3\cdot 2^n + 3\right) = 5^n - 3\cdot 4^n + 3\cdot 3^n - \cdot 2^n + 4.\end{aligned}$$

If $\alpha \in B(\mathcal{A}_0)$, then by definition of a set $B(\mathcal{A}_0)$ the quasinormal representation of a binary relation α has a form:

$$\alpha = \left(Y_4^\alpha \times Z_4\right) \cup \left(Y_3^\alpha \times Z_3\right) \cup \left(Y_1^\alpha \times Z_1\right) \cup \left(Y_0^\alpha \times \breve{D}\right),$$

where $Y_4^\alpha, Y_3^\alpha, Y_0^\alpha \in \{\varnothing\}$, or $Y_4^\alpha, Y_3^\alpha, Y_1^\alpha, Y_0^\alpha \in \{\varnothing\}$ are partitioning of the set X respectively;

$$\alpha = \left(Y_4^\alpha \times Z_4\right) \cup \left(Y_2^\alpha \times Z_2\right) \cup \left(Y_1^\alpha \times Z_1\right) \cup \left(Y_0^\alpha \times \breve{D}\right),$$

where $Y_4^\alpha, Y_2^\alpha, Y_1^\alpha \in \{\varnothing\}$, or $Y_4^\alpha, Y_2^\alpha, Y_1^\alpha, Y_0^\alpha \in \{\varnothing\}$ are partitioning of the set X respectively;

$$\alpha = \left(Y_3^\alpha \times Z_3\right) \cup \left(Y_2^\alpha \times Z_2\right) \cup \left(Y_1^\alpha \times Z_1\right) \cup \left(Y_0^\alpha \times \breve{D}\right)$$

where $Y_3^\alpha, Y_2^\alpha, Y_1^\alpha \in \{\varnothing\}$, or $Y_3^\alpha, Y_2^\alpha, Y_1^\alpha, Y_0^\alpha \in \{\varnothing\}$ are partitioning of the set X respectively;

$$\alpha = \left(Y_4^\alpha \times Z_4\right) \cup \left(Y_3^\alpha \times Z_3\right) \cup \left(Y_1^\alpha \times Z_1\right),$$

where $Y_4^\alpha, Y_3^\alpha \in \{\varnothing\}$, or $Y_4^\alpha, Y_3^\alpha, Y_1^\alpha \in \{\varnothing\}$ are partitioning of the set X respectively;

$$\alpha = \left(Y_4^\alpha \times Z_4\right) \cup \left(Y_2^\alpha \times Z_2\right) \cup \left(Y_0^\alpha \times \breve{D}\right),$$

where $Y_4^\alpha, Y_2^\alpha \in \{\varnothing\}$, or $Y_4^\alpha, Y_2^\alpha, Y_0^\alpha \in \{\varnothing\}$ are partitioning of the set X respectively;

$$\alpha = \left(Y_3^\alpha \times Z_3\right) \cup \left(Y_2^\alpha \times Z_2\right) \cup \left(Y_0^\alpha \times \breve{D}\right)$$

where $Y_3^\alpha, Y_2^\alpha \in \{\varnothing\}$, or $Y_3^\alpha, Y_2^\alpha, Y_0^\alpha \in \{\varnothing\}$ are partitioning of the set X respectively;

$$\alpha = \left(Y_2^\alpha \times Z_2\right) \cup \left(Y_1^\alpha \times Z_1\right) \cup \left(Y_0^\alpha \times \breve{D}\right),$$

where $Y_2^\alpha, Y_1^\alpha \in \{\varnothing\}$, or $Y_2^\alpha, Y_1^\alpha, Y_0^\alpha \in \{\varnothing\}$ are partitioning of the set X respectively.

Of this and from the equality (3), (4) and (5) follows that

$$\begin{aligned}\left|B(\mathcal{A}_0)\right| &= 4\cdot\left(2^n - 2\right) + 7\cdot\left(3^n - 3\cdot 2^n + 3\right) + 3\cdot\left(4^n - 4\cdot 3^n + 6\cdot 2^n - 4\right) = \\ &= 4\cdot 2^n - 8 + 7\cdot 3^n - 21\cdot 2^n + 21 + 3\cdot 4^n - 12\cdot 3^n + 18\cdot 2^n - 12 = 3\cdot 4^n - 5\cdot 3^n + 2^n + 1.\end{aligned}$$

So, we have that:

$$\begin{aligned}|S_0| &= \left|B_0 \cup B(\mathcal{A}_0)\right| = \left(5^n - 3\cdot 4^n + 3\cdot 3^n - 2^n + 4\right) + \left(3\cdot 4^n - 5\cdot 3^n + 2^n + 1\right) = \\ &= 5^n - 2\cdot 3^n + 5, \\ |S_1| &= \left|B_0 \cup B(\mathcal{A}_0) \cup \{X\times Z_4, X\times Z_3, X\times Z_2\}\right| = 5^n - 2\cdot 3^n + 8.\end{aligned}$$

since $B_0 \cap B(\mathcal{A}_0) = B_0 \cap \{X\times Z_4, X\times Z_3, X\times Z_2\} = B(\mathcal{A}_0) \cap \{X\times Z_4, X\times Z_3, X\times Z_2\} = \varnothing$.

Theorem 2.2 is proved.

By symbol $\Sigma_{6.1}(X,5)$ we denoted all semilattices $D = \left\{Z_4, Z_3, Z_2, Z_1, \breve{D}\right\}$ of the class $\Sigma_6(X,5)$ for which $Z_4 \cap Z_3 \cap Z_2 = \varnothing$. Of the last equality from the formal equalities of a semilattise D follows that $Z_4 \cap Z_3 \cap Z_2 = P_0 = \varnothing$, i.e. $|X| \geq 4$ since $P_4 \neq \varnothing$, $P_3 \neq \varnothing$, $P_2 \neq \varnothing$, $P_1 \neq \varnothing$.

In this case, the formal equalities of the semilattice D has a form:

$$\breve{D} = P_1 \cup P_2 \cup P_3 \cup P_4,$$
$$Z_1 = P_2 \cup P_3 \cup P_4,$$
$$Z_2 = P_1 \cup P_3 \cup P_4,$$
$$Z_3 = P_2 \cup P_4,$$
$$Z_4 = P_2 \cup P_3.$$

From the formal equalities of the semilattise D immediately follows, that:

$$P_4 = Z_3 \setminus Z_4,\ P_3 = Z_4 \setminus Z_3 = Z_1 \setminus Z_3,\ P_2 = Z_4 \cap Z_3 = Z_1 \setminus Z_2,\ P_1 = Z_2 \setminus Z_1.$$

In this case we suppose that $D \in \Sigma_{6.1}(X,5)$.

Let $D \in \Sigma_{6.1}(X,5)$. By symbols $\mathcal{A}_0$, $B(\mathcal{A}_0)$ and B_0 we denoted the following sets:

$$\mathcal{A}_0 = \left\{\left\{Z_4,Z_3,Z_1,\breve{D}\right\},\left\{Z_4,Z_2,Z_1,\breve{D}\right\},\left\{Z_3,Z_2,Z_1,\breve{D}\right\},\left\{Z_4,Z_3,Z_1\right\},\left\{Z_4,Z_2,\breve{D}\right\},\left\{Z_3,Z_2,\breve{D}\right\},\left\{Z_2,Z_1,\breve{D}\right\}\right\},$$
$$B(\mathcal{A}_0) = \left\{\alpha \in B_X(D) \mid V(X^*,\alpha) \in \mathcal{A}_0\right\},\ B_0 = \left\{\alpha \in B_X(D) \mid V(X^*,\alpha) = D\right\}$$

Remark, that the of the sets B_0 and $B(\mathcal{A}_0)$ are external elements for the semigroup $B_X(D)$.

Theorem 3.1. *Let $D \in \Sigma_{6.1}(X,5)$, then the following statements are true:*

a) *if $\left|X \setminus \breve{D}\right| \geq 1$, then the $S_0 = B_0 \cup B(\mathcal{A}_0)$ is irreducible generating set for the semigroup $B_X(D)$;*

b) *if $X = \breve{D}$, then the $S_1 = B_0 \cup B(\mathcal{A}_0) \cup \{X \times Z_4, X \times Z_3, X \times Z_2\}$ is irreducible generating set for the semigroup $B_X(D)$.*

Theorem 3.2. *Let $D = \left\{Z_4, Z_3, Z_2, Z_1, \breve{D}\right\} \in \Sigma_{6.0}(X,5)$ and*

$$\mathcal{A}_0 = \left\{\left\{Z_4,Z_3,Z_1,\breve{D}\right\},\left\{Z_4,Z_2,Z_1,\breve{D}\right\},\left\{Z_3,Z_2,Z_1,\breve{D}\right\},\left\{Z_4,Z_3,Z_1\right\},\left\{Z_4,Z_2,\breve{D}\right\},\left\{Z_3,Z_2,\breve{D}\right\},\left\{Z_2,Z_1,\breve{D}\right\}\right\},$$
$$B(\mathcal{A}_0) = \left\{\alpha \in B_X(D) \mid V(D,\alpha) \in \mathcal{A}_0\right\};\ B_0 = \left\{\alpha \in B_X(D) \mid V(X^*,\alpha) = D\right\}.$$

Then the following statements are true:

a) *If $\left|X \setminus \breve{D}\right| \geq 1$, then the number $|S_0|$ elements of the set $S_0 = B_0 \cup B(\mathcal{A}_0)$ is equal to*

$$|S_0| = 5^n - 2 \cdot 3^n + 5.$$

b) *If $X = \breve{D}$, then the number $|S_1|$ elements of the set $S_1 = B_0 \cup B(\mathcal{A}_0) \cup \{X \times Z_4, X \times Z_3, X \times Z_2\}$ is equal to*

$$|S_1| = 5^n - 2 \cdot 3^n + 8.$$

Proof. The theorem 3.1 and 3.2 we may prove analogously of the theorems 2.1 and 2.2.

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