On Informational and Some Other n-Widths of Classes of Analytic Functions in a Disk

Sergii Vakarchuk, Mykhailo Vakarchuk

Alfred Nobel University, Dnipro

sbvakarchuk@gmail.com, mihailvakarchuk@gmail.com

Abstract

In the linear normed spaces $\widetilde{\mathcal{B}}(p,q,\lambda), 0 < p < q \le \infty, \min(q,\lambda) \ge 1$, the classes of functions analytic in the circle U_R, $R \ge 1$, are considered. These classes are defined by averaged moduli of continuity of second order and majorants. The exact values of the informational, linear and Bernstein n-widths of such classes are computed. The optimal method of recovery of functions from indicated classes are found too in the spaces $\widetilde{\mathcal{B}}(p,q,\lambda)$.

Key words: n-width, class of functions, moduli of continuity, majorant, optimal method of recovery.

1. Introduction.

Let $\mathbb{X}$ be a linear normed space and $\mathbb{B}$ the unit ball in it, $\mathcal{K}$ a convex centrally symmetric set from $\mathbb{X}$ containing null, $L_n \subset \mathbb{X}$ a linear n-dimensional subspace, $\Lambda: \mathbb{X} \to L_n$ a linear continuous operator from $\mathbb{X}$ into L_n, $\mathbb{Y}$ an arbitrary linear normed space containing $\mathcal{K}$, $\mathbb{Y}^*$ the dual space to $\mathbb{Y}$, $\{l_j\}_{j=1}^n \subset \mathbb{Y}^*$ an arbitrary set of linear functionals, $\mathbb{C}^n$ Cartesian product of n specimens of the set of complex number $\mathbb{C}$, $\mathrm{V}: \mathbb{C}^n \to \mathbb{X}$ a linear operator continuous from $\mathbb{C}^n$ into $\mathbb{X}$.

The quantity

$$i_n(\mathcal{K},\mathbb{X}) = \inf\left\{\inf\left\{\sup\left\{\left\|f - V\big(l_1(f),\dots,l_n(f)\big)\right\|_{\mathbb{X}}: f \in \mathcal{K}\right\}: \mathrm{V}\mathbb{C}^n \subset \mathbb{X}\right\}: \mathcal{K} \subset \mathbb{Y},\ \{l_j\}_{j=1}^n \subset \mathbb{Y}^*\right\}$$

is called the informational n-widths of a set $\mathcal{K}$ in $\mathbb{X}$ [1; ch.4, 4.1]. In fact, by analogy [2], a set of linear functionals $\{l_j\}_{j=1}^n \subset \mathbb{Y}^*$, where $\mathcal{K} \subset \mathbb{Y}$, participating in the determination of the quantity $i_n(\mathcal{K},\mathbb{X})$, can be considered as a coding method, which associates a function $f \in \mathcal{K}$ with information about it in the form of a point from $\mathbb{C}^n$, i.e. $\{z_1 = l_1(f),\dots,z_n = l_n(f)\} \in \mathbb{C}^n$. Then $\mathrm{V}: \mathbb{C}^n \to \mathbb{X}$ should be considered as a method of recovery of function f from $\mathcal{K}$ by a specified information. Thus, in the course of calculation two exact lower bounds in the formula for $i_n(\mathcal{K},\mathbb{X})$, we are dealing with finding a best method of recovery $\widetilde{\mathrm{V}}: \mathbb{C}^n \to \mathbb{X}$ (if it exists) of function f from $\mathcal{K}$, provided that an optimal linear normed space $\widetilde{\mathbb{Y}} \supset \mathcal{K}$ has already been found, in which $f \in \mathcal{K}$ is optimally coded by a set of linear functionals $\{\tilde{l}_j\}_{j=1}^n \subset \widetilde{\mathbb{Y}}^*$. From the above it follows that it appropriate to talk about an optimal method of $\left(\widetilde{\mathbb{Y}}, \{\tilde{l}_j\}_{j=1}^n \subset \widetilde{\mathbb{Y}}^*, \widetilde{\mathrm{V}}\right)$ - recovery (if it exists) of functions from a class $\mathcal{K}$ in a linear normed space $\mathbb{X}$. In this case we get

$$i_n(\mathcal{K},\mathbb{X}) = \sup\left\{\left\|f - \widetilde{\mathrm{V}}\left(\tilde{l}_1(f),\dots,\tilde{l}_n(f)\right)\right\|_{\mathbb{X}}: f \in \mathcal{K}\right\}. \quad (1)$$

Remind [3; ch.1, 1.1] that the quantity

$$\mathcal{E}(\mathcal{K},L_n)_{\mathbb{X}} \coloneqq \inf\{\sup\{\|f - \Lambda(f)\|_{\mathbb{X}}: f \in \mathcal{K}\}: \Lambda\mathbb{X} \subset L_n\}$$

characterizes the best linear approximation of a set $\mathcal{K}$ by elements from L_n. A linear operator $\widetilde{\Lambda}$ (if it exists) for which

$$\mathcal{E}(\mathcal{K},L_n)_{\mathbb{X}} \coloneqq \sup\left\{\left\|f - \widetilde{\Lambda}(f)\right\|_{\mathbb{X}}: f \in \mathcal{K}\right\}$$

defines the best linear approximation method for $\mathcal{K}$ in $\mathbb{X}$. The quantity

$$\delta_n(\mathcal{K},\mathbb{X}) = \inf\{\mathcal{E}(\mathcal{K},L_n)_{\mathbb{X}}: L_n \subset \mathbb{X}\} \quad (2)$$

is called the linear n-widths of a set $\mathcal{K}$ in $\mathbb{X}$. If there exists a subspace $\hat{L}_n \subset \mathbb{X}$, $\dim \hat{L}_n = n$, for which $\mathcal{E}(\mathcal{K},\hat{L}_n)_{\mathbb{X}} = \delta_n(\mathcal{K},\mathbb{X})$ then $\hat{L}_n$ is called extremal for the linear n-widths.

The quantity

$$b_n(\mathcal{K},\mathbb{X}) = \sup\{\sup\{\varepsilon > 0: \varepsilon\mathbb{B} \cap L_{n+1} \subset \mathcal{K}\}: L_{n+1} \subset \mathbb{X}\}$$

is named the Benrstein n-widths [4; ch.II, 1]. A subspace $\tilde{L}_{n+1} \subset \mathbb{X}$, $\dim \tilde{L}_{n+1} = n+1$, such that

$$b_n(\mathcal{K},\mathbb{X}) = \sup\{\varepsilon > 0: \varepsilon\mathbb{B} \cap \tilde{L}_{n+1} \subset \mathcal{K}\} \quad (3)$$

is called extremal the Bernstein n-widths. If $\mathbb{X}$ and $\mathbb{Y}$ are linear normed spaces and $\mathbb{Y}$ is subspace of $\mathbb{X}$, then from definition (3) the next inequality holds

$$b_n(\mathcal{K},\mathbb{X}) \geq b_n(\mathcal{K},\mathbb{Y}).$$

The following relationships are satisfied between approximation characteristics of a set $\mathcal{K}$ from $\mathbb{X}$

$$b_n(\mathcal{K},\mathbb{X}) \leq i_n(\mathcal{K},\mathbb{X}) \leq \delta_n(\mathcal{K},\mathbb{X}) \leq \mathcal{E}(\mathcal{K},L_n)_{\mathbb{X}} \quad (4)$$

Based on [2] and (1) – (4) we note that the problems of the exact calculation of the quantities of n-widths and search of optimal method of $\left(\tilde{\mathbb{Y}},\ \{\tilde{l}_j\}_{j=1}^{n} \subset \tilde{\mathbb{Y}}^*, \tilde{\mathbb{V}}\ \right)$ - recovery are equivalent in a certain sense. However, in the second problem, unlike first, it is necessary to find a set $\left(\tilde{\mathbb{Y}},\ \{\tilde{l}_j\}_{j=1}^{n} \subset \tilde{\mathbb{Y}}^*, \tilde{\mathbb{V}}\ \right)$ (if it exists) that guarantees the minimum possible recovery error of a class $\mathcal{K}$ in a space $\mathbb{X}$.

A disk of radius $R \in (0,\infty)$ and a set of analytic functions in it we will denote by symbols $U_R \coloneqq \{z \in \mathbb{C}: |z| < R\}$ and $\mathcal{A}(U_R)$ respectively in the complex plane. We denote $U \coloneqq U_1$ for $R = 1$. We will use a quantity

$$M_q(f,\rho) \coloneqq \begin{cases} \left(\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \left|f(\rho e^{it})\right|^q dt\right)^{1/q}, & 1 \leq q < \infty, \\ \max\left(\left|f(\rho e^{it})\right|: 0 \leq t < 2\pi\right), & q = \infty, \end{cases}$$

for a function f from $\mathcal{A}(U_R)$ and $\rho \in (0,R)$. By symbol $H_{q,R}$, $1 \leq q \leq \infty$, $R \in (0,\infty)$, we denote the Hardy space consisting of functions $f \in \mathcal{A}(U_R)$ that have a finite norm

$$\|f\|_{H_{q,R}} = \lim\{M_q(f,\rho): \rho \to R - 0\}.$$

We put $H_q \coloneqq H_{q,1}$ in case $R = 1$.

For an arbitrary function $f \in H_{q,R}$ we write the second order moduli of continuity

$$\omega_2(f,2t)_{H_{q,R}} \coloneqq \sup\left\{\left\|f(ze^{i\tau}) - 2f(z) + f(ze^{-i\tau})\right\|_{H_{q,R}}: |\tau| \leq t\right\}, t \geq 0.$$

We assume that $\Psi(t), t \geq 0$, is an arbitrary continuous increasing function such that $\Psi(0) = 0$. We will call it a majorant. Let us consider the classes of functions

$$W_{H_{q,R}}(\omega_2,\Psi;\mu) \coloneqq$$

$$\left\{f \in H_{q,R}: \frac{1}{t}\int_0^t \omega_2(f,2\tau)_{H_{q,R}}\left(1 + (\mu^2 - 1)\sin\left(\frac{\pi\tau}{2t}\right)\right)d\tau \leq \Psi(t)\ \forall t \in \left(0,\frac{\pi}{2}\right)\right\}, \quad \mu \geq \frac{1}{2}, \quad (5)$$

which are defined by the averaged second order moduli of continuity in $H_{q,R}$ and majorants.

The classes of functions (5) were introduced by N. Ainulloev in [5] for $R = 1$ and $1 \leq q \leq \infty$. The Kolmogorov n-widths were computed for them in the Hardy spaces H_q, $1 \leq q \leq \infty$, when a majorant Ψ satisfies the condition

$$\frac{\Psi(t)}{\Psi(u)} \geq \frac{\pi}{\pi-2}\int_0^1 \left(1-\cos\left(\frac{\pi t x}{2u\mu}\right)\right)_* \left(1+(\mu^2-1)\sin\left(\frac{\pi x}{2}\right)\right)dx \qquad (6)$$

for a given $\mu \geq 1/2$ and for any values $u \in (0,\ \pi/2]$ and $t \in (0,\infty)$. Here

$$(1-\cos x)_* := \{1-\cos x \text{ for } 0 \leq x \leq \pi;\ 2 \text{ for } x > \pi\}.$$

Let us denote by $f^{(m)}, m \in \mathbb{N}$, the m-th order derivative of a function $f \in \mathcal{A}(U_R)$ by complex variable z, i.e.

$$f^{(m)}(z) = \sum_{k=m}^{\infty} \alpha_{k,m} c_k(f) z^{k-m},$$

where $\alpha_{k,m} := k(k-1)\dots(k-m+1)$, $c_k(f)$ are Taylor's coefficients of a function f. For the case $m=0$ we assume that $f^{(0)} = f$ and $\alpha_{k,0} := 1\ \forall k \in \mathbb{N}$. We will consider the classes of functions

$$W^m_{H_{q,R}}(\omega_2,\Psi;\mu) := \left\{f \in \mathcal{A}(U_R): f^{(m)} \in W_{H_{q,R}}(\omega_2,\Psi;\mu)\right\},$$

where $R \geq 1$, $m \in \mathbb{Z}_+$. If $m=0$ then the class $W^0_{H_{q,R}}(\omega_2,\Psi;\mu)$ is (5).

By the symbol $\widetilde{\mathcal{B}}(p,q,\lambda), 0<p<q\leq\infty, \min(q,\lambda) \geq 1$, we denote the linear normed space of complex-valued functions $f(z), z \in U$, that have a finite norm

$$\|f\|_{\widetilde{\mathcal{B}}(p,q,\lambda)} := \left\{\int_0^1 (1-\rho)^{\lambda(1/p-1/q)-1}\left(\frac{1}{2\pi}\int_0^{2\pi} \left|f\left(\rho e^{i\tau}\right)\right|^q d\tau\right)^{\lambda/q} d\rho\right\}^{1/\lambda}.$$

Here the integrals are understood in the Lebesque sense. The set of all analytic functions $f \in \mathcal{A}(U)$ for which $\|f\|_{\widetilde{\mathcal{B}}(p,q,\lambda)} < \infty$ we will denote by $\mathcal{B}(p,q,\lambda)$. The spaces $\mathcal{B}(p,q,\lambda)$, $0<p<q\leq\infty$, $\min(q,\lambda) \geq 1$, are Banach's and [6]

$$\|f\|_{p,q,\lambda} := \|f\|_{\mathcal{B}(p,q,\lambda)} = \begin{cases}\left(\int_0^1 (1-\rho)^{\lambda(1/p-1/q)-1} M_q^{\lambda}(f,\rho)d\rho\right)^{1/\lambda}, & 1 \leq \lambda < \infty,\\ \sup\left((1-\rho)^{1/p-1/q} M_q(f,\rho): 0<\rho<1\right), & \lambda = \infty.\end{cases}$$

Extremal problems of the approximation theory in the linear normed space $\widetilde{\mathcal{B}}(p,q,\lambda)$ and in the Banach space $\mathcal{B}(p,q,\lambda)$ of functions analytic in the unit disk were considered respectively in the work [7] and works [8] – [10].

2. Methodology of reasoning

It is clear from (4) that for the exact calculation of the informational n-width it is necessary to obtain the lower bound for the Bernstein n-width and upper bound for the linear n-width of a class of function and these bounds must be equal. Some principle moments of the proof of main results from [7] and [11] were used too.

The main result. Let us introduce the necessary notation for the formulation of main result of this communication.

We denote by

$$t_* := \frac{\pi}{2(n-m)\mu}, \qquad \text{where} \qquad \mu \geq \frac{1}{2}, \qquad n \in \mathbb{N}, \qquad m \in \mathbb{Z}_+, \qquad n > m;$$

$$\beta_{k,m} := \frac{2(n-m)}{\pi-2}\int_0^{t_*} \left(1-\sin\left((n-m)\tau\mu\right)\right)\cos\left((k-m)\tau\right)d\tau;$$

for $n, m \in \mathbb{N}$ and $n > m$

$$\psi_{k,m}(z) := \begin{cases} z^k, & k = \overline{0, m-1} \\ \left[R^{2(n-k)} + \frac{\alpha_{k,m}}{\alpha_{2n-k,m}}\left(\beta_{k,m}\left(1 - \left(\frac{k-m}{2n-k-m}\right)^2\right) - 1\right)|z|^{2(n-k)}\right] z^k, & k = \overline{m, n-1}, \end{cases}$$

and for $m = 0,\ n \in \mathbb{N}$

$$\psi_{k,0}(z) := \left[R^{2(n-k)} + \left(\beta_{k,0}\left(1 - \left(\frac{k}{2n-k}\right)^2\right) - 1\right)|z|^{2(n-k)}\right] z^k, \qquad k = \overline{0, n-1},$$

where $R \geq 1, z \in U_R$;

$$\mathcal{H}_{p,q,\lambda}(n) := \{B^{1/\lambda}(n\lambda + 1, \lambda(1/p - 1/q)) \text{ for } 1 \leq \lambda < \infty;\ \mathcal{F}_{p,q}(n) \text{ for } \lambda = \infty\},$$

where $B(a,b), a > 0, b > 0$, is Euler integral of the first kind and

$$\mathcal{F}_{p,q}(n) := (1 + n/(1/p - 1/q))^{1/q - 1/p}(1 + (1/p - 1/q)/n)^{-n}, 0 < p < q < \infty.$$

Note that the value $\mathcal{H}_{p,q,\lambda}(n) \to 0$ for $n \to \infty$.

Theorem. Let $\mu \geq 1/2$ be an arbitrary fixed number; $R \geq 1;\ 0 < p < q < \infty, \min(q, \lambda) \geq 1; n \in \mathbb{N}, m \in \mathbb{Z}_+$ and $n > m$; a majorant Ψ satisfies the condition (6). Then the following equalities hold:

$$b_n\left(W^m_{H_{q,R}}(\omega_2, \Psi; \mu), \mathcal{B}(p,q,\lambda)\right) = h_n\left(W^m_{H_{q,R}}(\omega_2, \Psi; \mu), \widetilde{\mathcal{B}}(p,q,\lambda)\right) =$$

$$\mathcal{E}\left(W^m_{H_{q,R}}(\omega_2, \Psi; \mu), \hat{L}_{n,m}\right)_{\widetilde{\mathcal{B}}(p,q,\lambda)} = \sup\left\{\|f - \tilde{\Lambda}_{n-1,m}(f)\|_{\widetilde{\mathcal{B}}(p,q,\lambda)} : f \in W^m_{H_{q,R}}(\omega_2, \Psi; \mu)\right\} =$$

$$\frac{R^{m-n}}{\alpha_{n,m}} \mathcal{H}_{p,q,\lambda}(n)\, \Psi\left(\frac{\pi}{2(n-m)\mu}\right),$$

Where $h_n\left(W^m_{H_{q,R}}(\omega_2, \Psi; \mu), \widetilde{\mathcal{B}}(p,q,\lambda)\right)$ is anyone from n-widths: i_n, δ_n or b_n. Furthermore

1) in the case of linear n-width $\delta_n\left(W^m_{H_{q,R}}(\omega_2, \Psi; \mu), \widetilde{\mathcal{B}}(p,q,\lambda)\right)$ the subspace $\hat{L}_{n,m} := \operatorname{span}\{\psi_{k,m}(z)\}_{k=0}^{n-1}$ is extremal;

2) the function

$$\tilde{\Lambda}_{n-1,m}(f,z) := \sum_{k=0}^{n-1} \eta_{k,m} c_k(f) \psi_{k,m}(z),$$

where $\eta_{k,m} := \{1 \text{ for } k = \overline{0, m-1};\ R^{2(k-n)} \text{ for } k = \overline{m, n-1}\}$ when $m \in \mathbb{N}$ and $\eta_{k,0} := R^{2(k-n)}$, $k = \overline{0, n-1}$, when $m = 0$, is the best linear approximation method for the class $W^m_{H_{q,R}}(\omega_2, \Psi; \mu)$ in the space $\widetilde{\mathcal{B}}(p,q,\lambda)$;

3) the exact value of the informational n-width $i_n\left(W^m_{H_{q,R}}(\omega_2, \Psi; \mu), \widetilde{\mathcal{B}}(p,q,\lambda)\right)$ is realized in the linear normed space $\widetilde{\mathcal{B}}(p,q,\lambda)$ by the optimal method of $\left(\widetilde{\mathbb{Y}} := H_q, \{\tilde{l}_j(f) := c_{j-1}(f)\}_{j=1}^n \subset \widetilde{\mathbb{Y}}^*, \widetilde{\mathbb{V}} := \tilde{\Lambda}_{n-1,m}(f)\right)$ – recovery of a function f from the class $W^m_{H_{q,R}}(\omega_2, \Psi; \mu)$;

4) for the Bernstein n-width $b_n\left(W^m_{H_{q,R}}(\omega_2, \Psi; \mu), \mathcal{B}(p,q,\lambda)\right)$ the subspace $\tilde{L}_{n+1} := \operatorname{span}\{z^k\}_{k=0}^n$ is extremal.

3. Discussion

Why the informational n-width is necessary? The calculation of such characteristic for a class of functions in a linear normed space give us, for example, the answers on the questions:

- what is the optimal method of recovery of functions from this class,
- what is a minimal number of linear functionals which are sufficiently that to recover 0f functions from this class with given accuracy, ets.

The answers to these questions we can find from the main result of this massage for the considered classes of functions.

References:

1. K.Yu. Osypenko, Optimal recovery of analytic functions, Nova Science Publishers, Inc., Huntington, New York (2000).
2. N.P. Korneichuk, Widths in L_p of classes of continuous and of differentiable functions, and optimal methods of coding and recovering functions and their derivatives, Math. USSR-Izv., 1982; 18, № 2, 227-247 (1982).
3. N.P. Korneichuk, Extremal problems of approximation theory, Nauka, Moskow (1976).
4. A. Pinkus, n-Widths in approximation theory, Springer – Verlag, Heidelberg (1985).
5. N. Ainulloev, Widths of classes of analytic functions in the unit disk. In the collection of scientific works: Geometric problems of the theory of functions and sets. Kalinin, 91–101 (1986).
6. M.I. Gvaragze, On a class of spaces of analytic functions, Math. Notes, 21, № 2, 79–84 (1977).
7. S.B. Vakarchuk, Best linear methods of approximation and widths of classes of analytic functions in a disk, Math. Notes, 57, № 1-2, 21–27 (1995).
8. S.B. Vakarchuk, On the best polynomial approximation in some Banach spaces for functions analytic in the unit disk, Math. Notes, 55, № 4, 338–343 (1994).
9. S.B. Vakarchuk, On some extremal problems of approximation theory in the complex plane, Ukr. Math. J., 56, № 9, 1371–1390 (2004).
10. S.B. Vakarchuk, M.B. Vakarchuk, K-functionals and extreme problems of the approximation theory for classes of analytic functions in a circle. I., Ukr. Math. J., 74, № 4, 532–550 (2022).
11. S.B. Vakarchuk, M.Sh. Shabozov, The widths of classes of analytic functions in a disk. Sbornik Math., 201, № 8, 1091–1110 (2010).