Limit theorems for age-dependent branching process with emigration

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Abstract

In this paper we consider age-dependent branching processes with emigration as a part of the same process. We prove limit theorems for the process that describe number of particles which emigrated.

Keywords. branching processes, stochastic additive functional, critical branching processes, emigration, perron root, moments.

1. Introduction

We will define multi-type age –dependent branching process with emigration by analogy with [2].

As in the multi-type process without emigration we have n types of particles $T_1, T_2, \ldots, T_n$. Each T_i-th type particle has random duration of 'presence' in process τ_i with distribution function

$$P(\tau_i \le t) = G^i(t), G^i(0+) = 0$$

We will assume that $G^i(t)$ are absolutely continuous.

At the end of its presence particle of any type is transformed into arbitrary number of particles of any types or emigrate. And let's define type T_0 as a type of particles which emigrated. Conditional probability (if transformation took place when the age attained by the original particle was u) $p_\alpha^i(u)$ of transformation into a set consisting of α_i $T_i -$th type particles, $i = \overline{0,n}),$ where $\alpha = (\alpha_0, \alpha_1, \alpha_2, ..\alpha_n)$ is $n+1$-dimensional vector and α_0 can be 0 (in this case other components of vector take non-negative integer values) or 1 (in this case other components of vector are zeroes). Evolution of particle is defined by joint distribution of random variable τ_i and random vector $v_i = (v_i^0, v_i^1, \ldots, v_i^n),$ which characterize progeny of this particle.

$$P(\tau_i \in B, v_i = \alpha) = \int_B p_\alpha^i(u) dG^i(u)$$

Particle of type T_0 at any moment transforms into itself (or doesn't transform, you can interpret it either way), so it can be viewed as T_0-th type particle has degenerate at zero life duration distribution function ($G^0(t) = 1, \forall t > 0).$ Also we will assume that $p_1^0(u) = 1, p_\alpha^0(u) = 0$ for all $u \ge 0$ and all vectors $\alpha \ne (1,0,\ldots,0),$ where $1 = \underbrace{(1,0,\ldots,0)}_{n+1}.$ Vector $\mu_i(t) = \left(\mu_i^0(t), \mu_i^1(t), \ldots, \mu_i^n(t)\right)$ denotes number of particles of types $T_0, T_1, T_2, \ldots, T_n$ at the moment *t*, under the condition, that at initial moment there existed one $T_i -$ *th* type particle. And let's also assume, that vector are right continuous.

Let's denote by $P^i(*)$ conditional probability, under condition that at initial moment of time there existed one particle of type T_i.

Let's also introduce generating functions

$$h^i(t,s) = \sum_\alpha p_\alpha^i(t) s^\alpha \text{ and } F^i(t,s) = \sum_\alpha P^i(\mu_i(t) = \alpha) s^\alpha, i = \overline{0,n},$$

$$s = (s_0, s_1, \ldots, s_n), s^\alpha = s_0^{\alpha_1}, s_1^{\alpha_0}, \ldots., s_0^{\alpha_n},$$

$$F(t,s) = (F^0(t,s), F^1(t,s), \ldots., F^n(t,s)), h(t,s) = (h^0(t,s), h^1(t,s), \ldots., h^n(t,s)).$$

It's clear that $F^0(t,s) = h^0(t,s) = s_0$ for all t.

Lemma 1. Generating functions $F^i(t,s)$ satisfy with $s \le 1$ and $t \ge 0$ next system of integral equations:

$$F^i(t,s) = \int_0^t h^i(u, F(t-u,s))dG^i(u) + s_i(1 - G^i(t)), i = \overline{0,n}. \quad (1)$$

Proof of this lemma is analogical to proof of the theorem 1 [2], p.234.

2. Moments

During the paper we will use following notations:

$$\alpha^i_j(u) = \left.\frac{\partial h^i(u,s)}{\partial s_j}\right|_{s=1}, \; A^i_j = \int_0^\infty \alpha^i_j(u)dG^i(u),$$

$$b^i_{jk}(u) = \left.\frac{\partial h^i(u,s)}{\partial s_j \partial s_k}\right|_{s=1}, \; B^i_{jk} = \int_0^\infty b^i_{jk}(u)dG^i(u),$$

$$A^l_{\rho k} = \int_0^\infty ue^{-\rho u} a^l_k(u)dG^l(u), \quad B^l_{\rho jk} = \int_0^\infty e^{-2\rho u} b^l_{jk}(u)dG^l(u),$$

$$M^k = \int_0^\infty udG^k(u), \quad M^{lk}_a = \int_0^\infty ua^l_k dG^l(u),$$

$$B = \sum_{l,k,m=1}^n B^l_{mk} v^l u^k u^m, \quad M_a = \sum_{l,k=1}^n M^{lk}_a v^l u^k, \quad M_0 = \sum_{l=1}^n v^l \int_0^\infty a^l_0(u)dG^l(u),$$

where ρ denotes such number, that perron root of matrix $e^{-\rho l}\|A^i_j\|_{i,j=\overline{1,n}}$ equals to one, $u_\rho = (u^1_\rho, u^2_\rho, \ldots, u^n_\rho)$ and $v_\rho = (v^1_\rho, v^2_\rho, \ldots, v^n_\rho)$ denote right and left eigenvectors of this matrix, $(v_\rho, 1) = \sum_{k=1}^n v^k_\rho = 1$ (in critical case $\rho = 0$ and we denote $u_\rho = u, v_\rho = v$).

We will call described above branching process (b.p) ξ. We will also introduce additional b.p. ξ', which differ from ξ only by the fact, that conditioned probability of transforming into the empty set $p^{i'}_0(u)$ equals to the sum of $p^i_0(u)$ and $p^i_1(u)$ where $0 = \underbrace{(0,0,\ldots,0)}_{n+1}$. So process ξ' could be seen simply as a multi-type age dependent process. We will call process ξ subcritical (critical or supercritical) iff ξ' is subcritical (critical or supercritical) (see, for example, [2]. It's clear that moments $A^{i'}_j$ of process ξ', for $i, j = \overline{1,n}$, and it's also clear that $a^i_0(u) = p^i_1(u)$.

Now consider matrix $A = \|A^i_j\|_{i,j=\overline{0,n}}$ of moments of process ξ

Obviously $A^0_0 = 1$ and $A^0_j = 0 \quad \forall j = \overline{1,n}$. Then has A from $\begin{bmatrix} 1 & 0 \\ A_{10} & A' \end{bmatrix}$, where A' is matrix of moments of process ξ', $A_{10} = (A^1_0, A^2_0, \ldots, A^n_0), 1 = 1, 0 = \underbrace{(0,0,\ldots,0)}_{n}$.

3. Supercritical case.

As we mentioned above, our b.p. is a special type of decomposable branching processes, but it turns out that we could apply theory of stochastic additive functionals from b.p. (see [3]) to our case. In fact we can interpret processes $\mu_0^i(t)$ where amount of product $\gamma(t)$ (which depends on type of particle, age at the moment *t* and life duration (we called it presence above)), produced by 1 particle during the time period $0 \le t \le \tau$, (τ as usually denote life duration) is

$$\gamma(t) = \begin{cases} 1, \text{ if particle emigrates at moment } \tau, \\ 0, \text{ otherwise}, \end{cases}$$

Analogy of theorem 2 (and corollary 1) [3] takes place, which also can be proven using slight modification of results obtained in [1], if we approach this as decomposable b.p.:

Theorem 1. If process ξ is supercritical, $A_{\rho k}^l(u)$ and $B_{\rho jk}^l$ are finite, $i, j, k = \overline{1,n}$, than random variables (r.v.) $\frac{e^{-\rho t}\mu_0^i(t)}{K}, \frac{e^{-\rho t}\mu_1^i(t)}{K_1}, ..., \frac{e^{-\rho t}\mu_n^i(t)}{K_n}$ converge as $t \to \infty$ in square mean to the same limit μ^i, where

$$K = \frac{\sum_{l=1}^{n} v_\rho^l \int_0^\infty e^{-\rho t} \int_0^t a_0^i(u) dG^i(u) dt}{\sum_{l,m=1}^{n} v_\rho^l u_\rho^m \int_0^\infty t e^{-\rho t} a_j^i(t) dG^l(t)}, \quad K_l = \frac{v_\rho^l \int_0^\infty e^{-\rho t} \int_0^t (1 - G^i(t) dt}{\sum_{l,m=1}^{n} v_\rho^l u_\rho^m \int_0^\infty t e^{-\rho t} a_j^i(t) dG^l(t)}, \quad l = \overline{1,n}$$

Furthermore, Laplace transform $\phi^l(s)$ of limit r.v. μ^l, satisfies equation

$$\phi^l(s) = \int_0^\infty h^l(u, \phi(se^{-\rho u})) dG^l(u), \tag{2}$$

with initial condition $\phi'(0) = u_\rho$, where $\phi(s) = (\phi^1(s), ..., \phi^n(s))$.

Corollary 1. If conditions of theorem 1 are satisfied, than distributions

$P^i\left(\frac{\mu_0^i(t)}{\sum_{k=1}^{n}\mu_k^i(t)} \le x \mid \sum_{k=1}^{n}\mu_k^i(t) > 0\right), i = \overline{1,n}$ converge to the degenerate distribution, localized at the point

$$\frac{\sum_{l=1}^{n} v_\rho^l \int_0^\infty e^{-\rho t} \int_0^t a_0^i(u) dG^i(u) dt}{\sum_{l=1}^{n} v_\rho^l \int_0^\infty e^{-\rho t} (1 - G^i(t) dt}.$$

4.Critical case.

In order to formulate theorem for critical case we will compare processes $\mu_0^i(t)$ with processes $N^i(t)$ − total number of particles born by the moment of time *t*, if at the moment $t = 0$ there existed one particle of type T_i.

Let $N_j^i(t)$ denote number of particles of type T_j, born by *t* , the $N^i(t) = \sum_{j=1}^{n} N_j^i(t)$. It is known [4] (in this paper is considered case with constant $p_\alpha^i(u) = p_\alpha^i$ for all *u*, but

$$E\left(exp\left\{i\sum_{j=1}^{n}\beta^j N_j^i(t)/v^j t^2\right\}|\sum_{j=1}^{n}\mu_j^i(t)\right)\xrightarrow{t\to+\infty}$$

$$\left(2B\sum_{j=1}^{n}\beta^j\right)^{1/2}/M_\alpha\left(sh\left(\left(2B\sum_{j=1}^{n}\beta^j\right)^{1/2}/M_\alpha\right)\right).$$

Then by letting $\beta^j = v^j\beta$, and since $\sum_{j=1}^{n} v^j = 1$, we get

$$E\left(exp\{i\beta^j N^i\ (t)/t^2\}|\sum_{j=1}^{n}\mu_j^i(t)\right)\xrightarrow{t\to+\infty}, \quad \rightleftarrows\rightleftarrows\ (2B\beta)^{1/2}/M_\alpha\left(sh\left((2B\beta)^{1/2}/M_\alpha\right)\right). \quad (3)$$

Also from [4] we can establish asymptotic behavior of moments $E^i(N_i(t))$ and $E^i(N_i^2(t))$:

$$E^i(N_i(t))\sim\frac{u^i t}{M_\alpha}, \quad E^i(N_i^2(t))\sim B\frac{u^i t^3}{3(M_\alpha)^3}.$$

Theorem 2. If the following conditions are satisfied:

i) integrals M^j, M_a^{jk}, B_{jk}^i are finite;

ii) $\int_0^\infty\int_t^\infty a_0^l(u)dG^l(u)dt = k_0^l < +\infty;$

iii) $\lim_{t\to\infty} t^2\int_t^\infty a_l^k(u)dG^l(u) < +\infty, \quad \lim_{t\to\infty} t^2(1-G^l(t)) < +\infty, l,k,j=\overline{1,n},$ then

$$E\left(\exp\{i\beta^j\mu_0^i(t)/t^2\}|\sum_{j=1}^{n}\mu_j^i(t)\right)\xrightarrow{t\to+\infty}, \quad (2BM_0\beta)^{1/2}/M_\alpha\left(sh\left((2BM_0\beta)^{1/2}/M_\alpha\right)\right).$$

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